

Equivariant Sheaf Neural Networks

A Unifying Framework for Equivariant Graph Neural Networks

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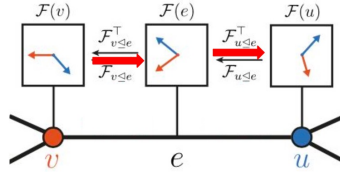
Introduction & Background

- Many physical systems are graphs embedded in the Euclidean space.
- Existing Equivariant GNN** preserve $E(n)$ symmetry but are often **isotropic**, struggling with shear, torsion, vorticity and directional interactions.
- Standard **Sheaf Neural Networks** model anisotropic transport but do not preserve $E(n)$ equivariance.



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How to move a vector x from v to u ?

Moving $x \in F(v) \rightarrow F(u)$ is done via maps composition!

$$F_{u \leq e}^T F_{v \leq e} x \in F(u)$$



What is a Sheaf Neural Network?

- A cellular sheaf equips a graph with local feature spaces $F(v), F(e)$ (**stalks**) and edge-dependent **transport maps**.
- The **disagreement** between neighbouring nodes is measured after transporting their representations into a common space.

$$(\delta_F h)_{j \rightarrow i} = F_{i \leq e} h_i - F_{j \leq e} h_j$$

- The **Sheaf Laplacian** is:

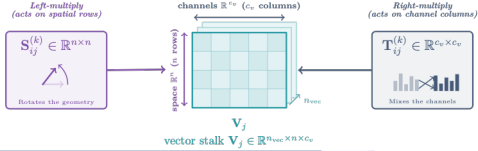
$$L_F(x)_u = \sum_{u, v \leq e} F_{u \leq e}^T (F_{u \leq e} x_u - F_{v \leq e} x_v)$$



Equivariant Sheaf Neural Networks

- Each node stalk contains **invariant scalar features** and $O(n)$ -**covariant vector features**.
- ESNN factorises every edge map into **covariant spatial transport** and **invariant channel mixing**.
- Restriction maps preserve rotations, reflections, and translations.**

$$\tau_{j \rightarrow i}(V_j) = \sum_{k=1}^K S_{ij}^{(k)} V_j T_{ij}^{(k)}$$



Research Question

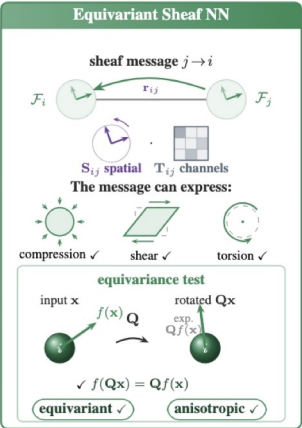
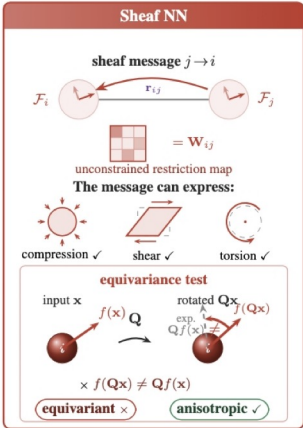
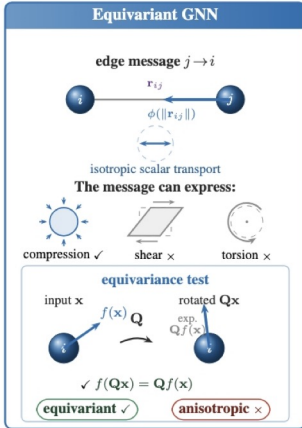
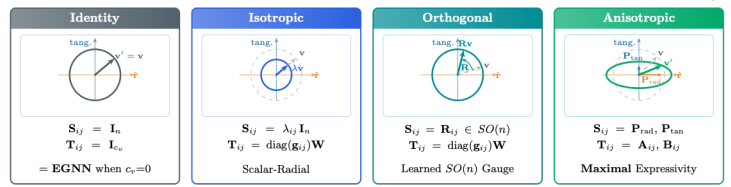
Can we combine anisotropic transport of SheafNNs with $E(n)$ -equivariance of geometric GNNs?



Transport Types

- Increasingly flexible constraints recover existing equivariant models and culminate in maximally expressive first-order anisotropic transport.

Directional Expressivity Increases $\rightarrow$

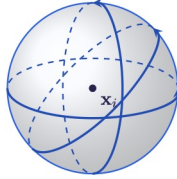


- Equivariant GNNs** preserve $E(n)$ -symmetry but usually rely on **isotropic radial interactions**.
- Standard **Sheaf Neural Networks** support **anisotropic transport**, but unconstrained restriction maps may violate equivariance.
- ESNN combines both properties by constraining spatial transport to be $O(n)$ -covariant.
- ESNN messages can model compression, shear, torsion, and direction-dependent interactions while remaining equivariant.

- Full $E(n)$ -equivariance is appropriate for isolated systems with no preferred direction.
- External fields, such as **gravity or uniform flow**, introduce a distinguished spatial axis.
- ESNN learns a relaxation parameter λ , preserving full equivariance when $\lambda \approx 0$.
- When the data require it, the model relaxes to cylindrical $O_g(n)$ -subequivariance, preserving only transformations that leave the field direction g fixed.

Full $E(n)$ Equivariance

Isotropic Symmetry



All spatial rotations are valid symmetries

Adaptive Symmetry Relaxation

Data-Driven Symmetry Breaking

Injection of directional bias via external field g

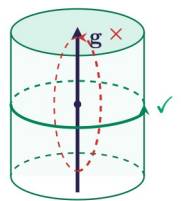
$$\lambda = 0 \text{ Learned } \lambda > 0$$

Strict $O(n)$ Relaxed $O_g(n)$

Adaptively relaxes symmetry driven by data geometry

Cylindrical $O_g(n)$

External Field

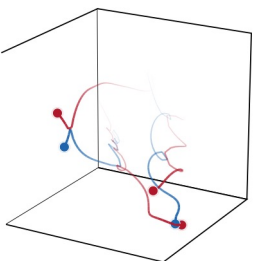


Transverse rotations break symmetry

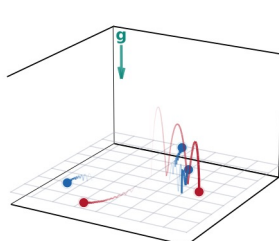


Preliminary Results

(a) N-Body (charged)



(b) N-Body Gravity



(a) N-body System

Method	MSE
Linear	0.0819
SE(3) Transformer	0.0244
Tensor Field Network	0.0155
Graph Neural Network	0.0107
Radial Field	0.0104
EGNN	0.0071
ESNN-Id	0.0055
ESNN-Iso	0.0058
ESNN-Ortho	0.0056
ESNN-Aniso	0.0052
PESNN-Id	0.0062
PESNN-Iso	0.0051
PESNN-Ortho	0.0051
PESNN-Aniso	0.0062

(b) N-body System with Gravity

Method	MSE
Linear	0.349
SE(3) Transformer	0.152
Tensor Field Network	0.097
Graph Neural Network	5.217
Radial Field	0.123
EGNN	0.108
ESNN-Id	0.073
ESNN-Iso	0.073
ESNN-Ortho	0.072
ESNN-Aniso	0.068
PESNN-Id	0.081
PESNN-Iso	0.076
PESNN-Ortho	0.069
PESNN-Aniso	0.067

- ESNN consistently improves over radial equivariant baselines.
- Anisotropic transport becomes the strongest choice when gravity introduces a preferred spatial axis.