Equivariant Sheaf Neural Networks

A Unifying Framework for Equivariant Graph Neural Networks

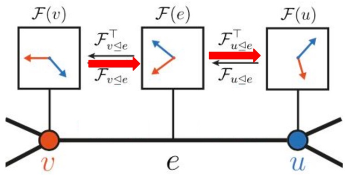
EEML
27 July / 1 August 2026
Cetinje, Montenegro

Alessio Borgi, Mario Severino, Fabrizio Silvestri, Pietro Liò

UNIVERSITY OF CAMBRIDGE
SAPIENZA UNIVERSITÀ DI ROMA
ellis UNIT CAMBRIDGE

Introduction & Background

- Many physical systems are graphs embedded in the Euclidean space.
- **Existing Equivariant GNN** preserve $E(n)$ symmetry but are often **isotropic**, struggling with shear, torsion, vorticity and directional interactions.
- Standard **Sheaf Neural Networks** model anisotropic transport but do not preserve $E(n)$ equivariance.



How to move a vector x from v to u?

Moving $\mathbf{x} \in \mathcal{F}(v) \to \mathcal{F}(u)$ is done via maps composition!

$$\mathcal{F}^{\top}_{u\trianglelefteq e}\mathcal{F}_{v\trianglelefteq e}\mathbf{x} \in \mathcal{F}(u)$$

What is a Sheaf Neural Network?

- A cellular sheaf equips a graph with local feature spaces $\mathcal{F}(v)$, $\mathcal{F}(e)$ (**stalks**) and edge-dependent **transport maps**.
- The **disagreement** between neighbouring nodes is measured after transporting their representations into a common space.

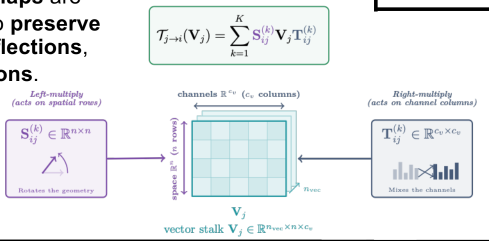
$$(\delta_{\mathcal{F}} h)_{j\to i} = \mathcal{F}_{i\trianglelefteq e} h_i - \mathcal{F}_{j\trianglelefteq e} h_j$$

- The **Sheaf Laplacian** is:

$$L_{\mathcal{F}}(\mathbf{x})_u = \sum_{u,v\trianglelefteq e} \mathcal{F}^{\top}_{u\trianglelefteq e}(\mathcal{F}_{u\trianglelefteq e}\mathbf{x}_u - \mathcal{F}_{v\trianglelefteq e}\mathbf{x}_v)$$

Equivariant Sheaf Neural Networks

- Each node stalk contains **invariant scalar features** and $\boldsymbol{O(n)}$**-covariant vector features**.
- ESNN factorises every edge map into **covariant spatial transport** and **invariant channel mixing**.
- **Restriction maps** are constrained to **preserve rotations, reflections,** and **translations**.

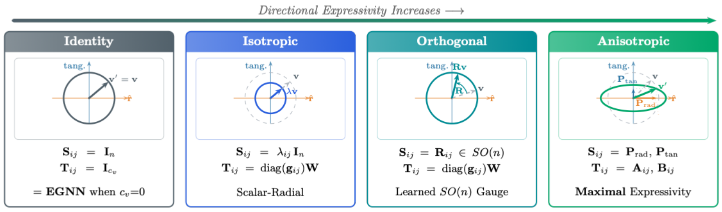


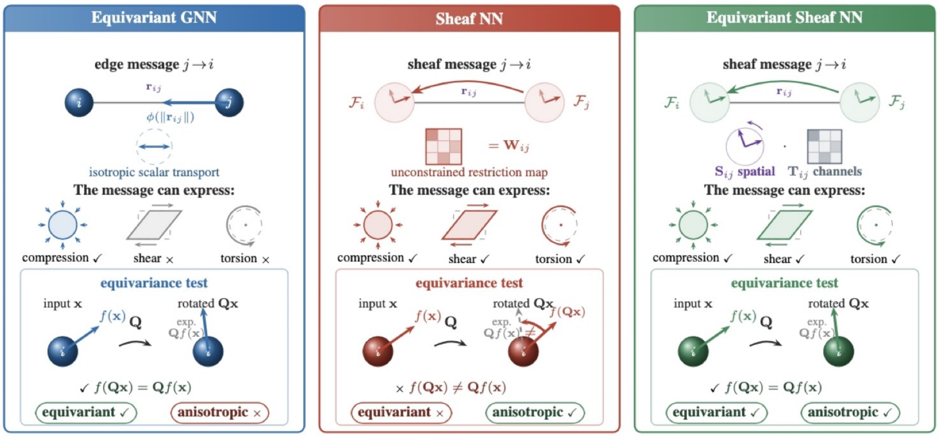
Research Question

Can we combine anisotropic transport of SheafNNs with $E(n)$-equivariance of geometric GNNs?

Transport Types

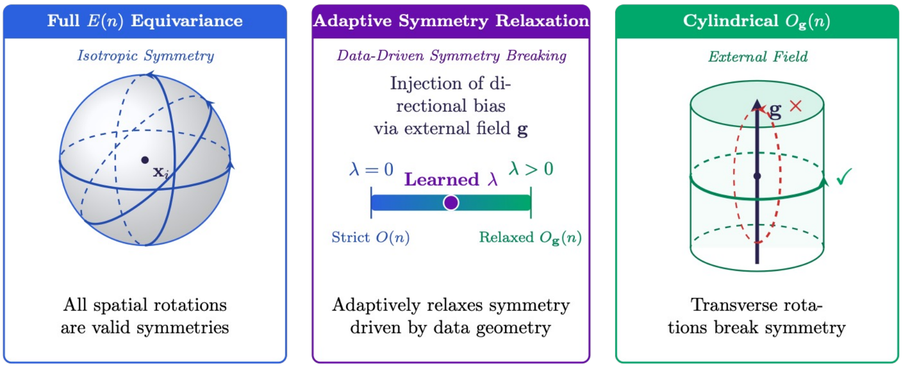
- Increasingly flexible constraints recover existing equivariant models and culminate in maximally expressive first-order anisotropic transport.





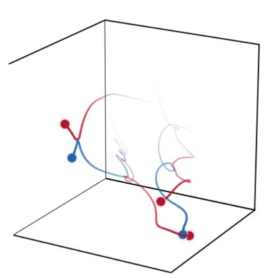
- **Equivariant GNNs** preserve $E(n)$-symmetry but usually rely on **isotropic radial interactions**.
- Standard **Sheaf Neural Networks** support **anisotropic transport**, but unconstrained restriction maps may violate equivariance.
- ESNN combines both properties by constraining spatial transport to be O(n)-covariant.
- ESNN messages can model compression, shear, torsion, and direction-dependent interactions while remaining equivariant.

- Full $E(n)$-equivariance is appropriate for isolated systems with no preferred direction.
- External fields, such as **gravity or uniform flow**, introduce a distinguished spatial axis.
- ESNN learns a relaxation parameter λ, preserving full equivariance when $\lambda \approx 0$.
- When the data require it, the model relaxes to cylindrical $O_g(n)$-subequivariance, preserving only transformations that leave the field direction g fixed.

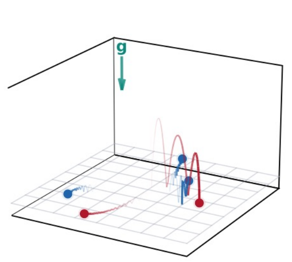


Preliminary Results

(a) N-Body (charged)



(b) N-Body Gravity



(a) N-body System

Method	MSE
Linear	0.0819
SE(3) Transformer	0.0244
Tensor Field Network	0.0155
Graph Neural Network	0.0107
Radial Field	0.0104
EGNN	0.0071
ESNN-Id	0.0055
ESNN-Iso	0.0058
ESNN-Ortho	0.0056
ESNN-Aniso	0.0052
PESNN-Id	0.0062
PESNN-Iso	0.0051
PESNN-Ortho	0.0051
PESNN-Aniso	0.0062

(b) N-body System with Gravity

Method	MSE
Linear	0.349
SE(3) Transformer	0.152
Tensor Field Network	0.097
Graph Neural Network	5.217
Radial Field	0.123
EGNN	0.108
ESNN-Id	0.073
ESNN-Iso	0.073
ESNN-Ortho	0.072
ESNN-Aniso	0.068
PESNN-Id	0.081
PESNN-Iso	0.076
PESNN-Ortho	0.069
PESNN-Aniso	0.067

- ESNN consistently improves over radial equivariant baselines.
- Anisotropic transport becomes the strongest choice when gravity introduces a preferred spatial axis.

*C. Bodnar, F. Di Giovanni, B. Chamberlain, P. Lio, and M. Bronstein. Neural Sheaf Diffusion: A topological perspective on heterophily and oversmoothing in GNNs. *NeurIPS*, 2022.